Error Fields, Resonances Injection and Extraction

We now introduce a method for arriving at approximate solutions to a wide variety of problems, which typically involve nonlinear behavior. This method was exploited and developed by Jurgen Moser for the solution of problems in Celestial Mechanics. First, we transform variables by a scale transformation. Then

\[ [s, x, z, t, p_s, p_x, p_z, H] \rightarrow [\theta, x, z, t, -K, p_x, p_z, U] \quad 9.1 \]

where \( \theta = s/R \), \( P_x = \frac{p_x}{P_0} \), \( P_z = \frac{p_z}{P_0} \), \( U = H/P_0 \), and \( -K = \frac{p_s R}{P_0} \). \( R = C/2\pi \), and \( P_0 \) is a constant reference momentum. Since \( P_x dx + P_z dz - U dt - K d\theta = 0 \) if \( p_s ds + p_x dx + p_z dz - H dt = 0 \), the transformed variables are canonical with \( K \) as the new Hamiltonian and \( \theta \) as the independent variable. The new Hamiltonian is (\( B\rho = P_0 c/e \))

\[
\frac{K}{R} = -\frac{(U - U_0)}{R\omega_0} + \left( \frac{1}{\gamma^2} - \alpha \right) \frac{(U - U_0)^2}{2RV_0 \omega_0} + \frac{(P_x^2 + P_z^2)}{2} + \left( \frac{\Omega^2 + \kappa}{2} \right) x^2 - \frac{\kappa z^2}{2} - \frac{A_{se}}{B\rho}. 
\]

Next we transform to angle action variables \( I_x, I_z, \sigma_x, \sigma_z \), via

\[
F_1 = -x^2 \left[ \tan \sigma_x + \alpha_x \right] + \frac{-z^2 \left[ \tan \sigma_z + \alpha_z \right]}{2\beta_x} - Ht 
\]

so that

\[
2I_x = \gamma x^2 + 2\alpha_x x P_x + \beta_x P_x^2, \quad \tan \sigma_x = \frac{\beta_x P_x + \alpha_x}{x} 
\]

and similarly for \( z \). The new Hamiltonian \( K' \) is

\[
K'/R = -\frac{(U - U_0)}{R\omega_0} + \left( \frac{1}{\gamma^2} - \alpha \right) \frac{(U - U_0)^2}{2RV_0 \omega_0} + \frac{I_x}{\beta_x} + \frac{I_z}{\beta_z} - \frac{A_{se}}{B\rho}. 
\]

Now we will consider only \( H = H_0 \) and ignore energy motion. \( K' \) above is not a constant of the motion, even if \( A_{se} = 0 \), because the \( \beta \)'s depend on \( \theta \). To improve matters, we make a further transformation to new phase variables \( \phi_x \) and \( \phi_z \) by

\[
F_3 = I_x \left[ \sigma_x + \int_0^{\theta R d\theta} - \nu_x \right] + I_z \left[ \sigma_z + \int_0^{R d\theta} - \nu_z \right] 
\]

Then \( K'' = I_x \nu_x + I_z \nu_z + R \frac{A_{se}}{B\rho} \). If \( A_{se} = 0 \), \( K'' \) is a constant of the motion. Then:

\[
x = \sqrt{2\beta_x I_x} \left[ \cos(\phi_x + \psi_x) \right], \quad P_x = \sqrt{2I_x} \left[ \sin(\phi_x + \psi_x) - \alpha_x \cos(\phi_x + \psi_x) \right] 
\]
\[ \sigma_x = \phi_x + \psi_x, \quad \psi_x = \frac{\theta}{\beta_x} \int_0^{Rd\theta} + \psi_x \theta \]

Similarly for \( z \). The \( \psi \)'s are just the AG phase wiggle about the mean phase advance. A large class of error fields are due to a vector potential \( A_s \) without any components \( A_x, A_z \). The important cases we lose that way will be the field of a solenoid, and time dependent fields in accelerating cavities. We will return to these later. In addition we only consider errors in straight sections of the orbit, not curved sections. In straight sections the coordinate systems are Cartesian, and the potentials can be represented by simple polynomials, whereas in curved sections the coordinates are curvilinear and the potentials are more complex. For example just as there was an \( x^2 \) term in the potential of a dipole, there will be an \( x^3 \) term in the field of a gradient magnet, etc. As it develops this does not change the development of the method to approach these problems, but it might change the values of the resultant quantities.

Then the fields to be considered are two dimensional, and since they satisfy the Laplace equation, both the real and imaginary parts of any analytic function of \( x+iz \) are solutions. Since the solutions must be regular at \((0,0)\), the fields can be represented by power terms \((x+iz)^n\) where \( n \) is a positive integer. The terms so obtained are usually called "multipoles" of order \( n \), or \( 2n \)-poles. Thus \( n = 3 \) is called a "sextupole". The dipole field is in the \( z \) direction, as is the main quadrupole field at \( z = 0 \) as described previously. Such multipoles are called "normal" while those with \( B_z = 0 \) on \( z = 0 \) are called "skew" multipoles. It is straightforward to show that the normal multipoles correspond to the choice of the real part of \((x+iz)^n\) for \( A_s \) and the skew multipoles correspond to the choice of the imaginary part of \((x+iz)^n\) for \( A_s \).

We could as easily have chosen to represent the magnetic field by a scalar potential \( \Phi(x, z) \). We recall from potential theory that for the two dimensional field the function \( A_s + i\Phi \) is an analytic function of \( x+iz \). Then the choice of real and imaginary is simply reversed to get the scalar potential from \((x+iz)^n\) for normal and skew multipoles.

The term in the Hamiltonian for one multipole becomes

\[
B^{n-1} \sum_{j=0}^{2j} n! C_j (-1)^j z^{2j} x^{n-2j} \]

for a normal multipole, and

\[
B^{n-1} \sum_{j=1}^{2j+1} n! C_j (-1)^{j+1} z^{2j+1} x^{n-2j+1} \]

for skew multipoles. \( B^{n-1} \) is the \( n \)-1st derivative of \((B_x, B_z)\) with respect to \( x \) at \( z = 0 \) for (normal, skew) multipoles, and is in general a function of \( s \).

The error terms will be of the form \( B^{n-1} x^p z^q \) multiplied by a function of \( s \). We can expect that we will see systematic effects due to mean values of \( B^{n-1} \), and dynamic effects due to various Floquet-
Fourier harmonics of $B^{n-1}$. In particular if the dependence of $B^{n-1}$ is such that it tends to remain constant for long periods of time, instead of averaging to 0 rapidly, then we can expect to see large changes in $I_x$ or $I_z$ or both. This is similar to driving a pendulum or harmonic oscillator at its natural frequency. Usually such phenomena are called "resonant". (The "order" of the resonance driven by the term is $n = p + q$.) In general, only one of the terms is resonant. Then we are justified in neglecting all the others, since they average out quickly in the motion. When this is not true, the approximation breaks down, and we have, in general, stochastic (unpredictable, or "random") motion.

We will want to pursue some systematic effects, so we will consider off-momentum orbits for these. In these cases we will substitute in the potential

$$x = x_p \frac{\Delta P}{P} + \sqrt{2\beta_x} I_x \left[ \cos(\phi_x + \psi_x) \right]$$

Dipole Errors

To pursue matters further, let us consider the introduction of an error dipole field, say $A_s = B_0(\theta) x$, corresponding to a field in the z direction. Since the field does not depend on z, the z motion will not be affected, so we can ignore it. The Hamiltonian becomes

$$K = I_x v_x + \frac{B_0 R}{B \rho} \sqrt{2\beta_x I_x} \left[ \cos(\phi_x + \psi_x) \right]$$

$$= I_x v_x + \frac{B_0 R}{B \rho} \sqrt{2\beta_x I_x} \left[ \frac{e^{i(\phi_x + \psi_x)} + e^{-i(\phi_x + \psi_x)}}{2} \right]$$

We make a Fourier expansion of the periodic coefficients of the $\phi_x$ terms.

$$\frac{B_0(\theta) R}{B \rho} \sqrt{2\beta_x(\theta)} \left[ e^{i\psi_x} \right] = \sum_{-\infty}^{\infty} a_k e^{ik\theta}$$

$$\frac{B_0(\theta) R}{B \rho} \sqrt{2\beta_x(\theta)} \left[ e^{-i\psi_x} \right] = \sum_{-\infty}^{\infty} a_k^* e^{-ik\theta}$$

and let $a_k = d_k e^{-i\mu_k}$. Then $K = I_x v_x + \sqrt{I_x} \sum_{-\infty}^{\infty} d_k \cos(\phi_x - k\theta - \mu_k)$.

Now if $v_x = p$ for some integer $p$ the phase of the $p^{th}$ term will vary slowly and we will have the condition for resonance. Then we ignore all the other terms. Next we make a transformation to rotating coordinates to remove the $\theta$ dependence of $K$ via the generating function $F_{2}(\phi_x, I_x') = I_x'(\phi_x - p\theta - \mu_p)$ so that the new coordinate $\gamma_x = \phi_x - p\theta - \mu_p$, and the new momentum is $I_x' = I_x$, while the new Hamiltonian is

$$K' = K + \frac{\partial F}{\partial \theta} = I_x'(v_x - p) + \sqrt{I_x} d_k \cos \gamma_x$$

which is an approximate constant of the motion.
\[
\frac{dI_x}{d\theta} = \frac{\partial K'}{\partial \gamma_x} = \sqrt{I_x} \frac{d}{dp} \sin(\gamma_x) \quad \frac{d\gamma_x}{d\theta} = (\nu_x - p) + \frac{\sqrt{I_x} \frac{d}{dp} \cos(\gamma_x)}{2} \tag{9.16}
\]

There is a fixed point at \( \gamma_x = 0, \sqrt{I_x} = \frac{d}{2(\nu_x - p)} \). The fixed point represents the motion of the "closed orbit" as it responds to the error fields \( B(\theta) \). Orbits (for \( K' = \text{constant} \)) are circles around the fixed point in a polar coordinate system \( (\sqrt{I_x}, \gamma_x) \). As the tune approaches the resonant value, the amplitude of the closed orbit oscillation grows without bound. This is called "integral resonance". Then the resonances driven by dipoles are;

\( \nu_x = \text{integer}: \) Normal dipole
\( \nu_z = \text{integer}: \) Skew dipole

This treatment only included the single harmonic closest to \( \nu_x \). Good practice demands that more precision be used for the closed orbit. Fortunately, since the equations are linear, solutions in closed form can always be found for this simple case. Practical tools for this problem are developed in Courant and Snyder.

Quadrupole Errors

We next consider the effects of error quadrupole fields. Here the normal and skew terms do not cause similar effects, and we must treat them separately. Then

\[
A_{\text{norm}} = \frac{B'}{2}(x^2 - z^2) \quad A_{\text{skew}} = B'_{s'xz} \tag{9.17}
\]

Considering the dependences of the \( B' \) and \( x^2 \) terms, the following resonances will be driven by quadrupoles;

\( 2\nu_x = \text{integer}: \) Normal quadrupole
\( \nu_x \pm \nu_z = \text{integer}: \) Skew Quadrupole
\( 2\nu_x = \text{integer}: \) Normal quadrupole

The normal terms, aside from a sign, will have similar effects in \( x \) and \( z \), so we will only treat one. The error term becomes

\[
\frac{B'}{2} x^2 = B'_{,x^2} I_x \cos^2 \left( \phi_x + \psi_x \right) = \frac{B'_{,x^2} I_x}{2} \left( 1 + \cos 2(\phi_x + \psi_x) \right) \tag{9.18}
\]

The first term has different \( \theta \) and \( \phi \) dependence from the second and is to be treated differently from the second. Due to it we find that

\[
\frac{dI_x}{d\theta} = 0, \quad \frac{d\gamma_x}{d\theta} = \nu_x + \frac{RB'_{,x^2}}{2B\rho} \tag{9.19}
\]

For the whole ring,

\[
\Delta\nu = \int_0^{2\pi} d\theta \frac{RB'_{,x^2}}{4\pi B\rho} = \int_0^c ds \frac{B'_{,x^2}}{4\pi B\rho} \tag{9.20}
\]
This is the same result obtained by matrix methods earlier. The second term can be expanded in Fourier series

\[ \sum_{b_k} b_k e^{i(2\phi_x - k\theta)} \]

Letting \( b_k = q_k e^{-i\chi_k} \) the Hamiltonian becomes

\[ K = I_x (\nu_x + \delta\nu_x) + I_x \sum_{-\infty}^{\infty} q_k \cos(2\phi_x - k\theta - \chi_k) \]

Now suppose that \( \nu_x + \delta\nu_x \approx r \) for some integer \( r \). Then only the \( r^{th} \) term in the sum will be slowly varying and we will neglect the others. We again perform a canonical transformation to remove the \( \theta \) dependence by the generating function

\[ F_2 = I_x' \left( \phi_x - \frac{r\theta - \chi_{xs}}{2} \right) \]

so that

\[ I_x' = I_x, \gamma_x = \phi_x - \frac{r\theta - \chi_{xs}}{2}, \text{ and} \]

\[ K' = I_x \left( \nu_x + \delta\nu_x - \frac{r}{2} + q_r \cos 2\gamma_x \right) + \frac{kI^2}{2} \]

is an approximate constant of the motion. We find

\[ \frac{dI_x}{d\theta} = I_x q_r \sin \gamma_x, \frac{d\gamma_x}{d\theta} = \nu_x + \delta\nu_x - \frac{r}{2} + q_r \cos \gamma_x \]

and we can solve for

\[ I_x = \frac{K'}{\left( \nu_x + \delta\nu_x - \frac{k}{2} \right) + q_r \cos \gamma_x} \]

We see that there is only one fixed point at \((0,0)\). When

\[ |q_r| > |\nu_x + \delta\nu_x - \frac{k}{2}|, \frac{d\gamma_x}{d\theta} \]

changes sign at \( \gamma_{xs} = \pm \arccos \frac{q_r}{\nu_x + \delta\nu_x - \frac{k}{2}} \). Then there are two separatrix rays from \((0,0)\) at \( \pm \gamma_{xs} \). These rays divide the phase space into two regions which do not communicate. In each region, orbits start at \( I_x = \infty \), come to a minimum value at \( \gamma_x = 0 \) or \( \pi \), and proceed to \( I_x = \infty \) again. This is called a "half integral resonance", or stopband, in that there is a finite region of \( \nu_x \) where there are no stable orbits.

For \( |q_r| < |\nu_x + \delta\nu_x - \frac{k}{2}|, \gamma_x \) transits the whole region of \( 2\pi \) while \( I_x \) oscillates in size. Then the picture that emerges as one approaches and transits a half integral resonance is the onset of beating of oscillation amplitude, followed by a stopband where all orbits are unstable, followed by beating again. As the stopband is traversed, the separatrices move from \( 0 \) to \( \pm \pi \) or vice versa, depending on the sign of
The "width" of the resonance is the width of the region in $\nu_x$ over which the stopband exists. It is clearly equal to $2|q_r|$.

**MOTION IN HALF-INTEGRAL STOPBAND: $Q = 1.5$**
MOTION IN HALF-INTEGRAL STOPBAND: \( Q = 1.5 \)

The quantity \( D \) is the ratio of stopband width \( q_r \) to the tune separation \( \nu \) from the resonance.

The skew component of quadrupole,

\[
A_{skew} = B'xz = B' \sqrt{2 \beta_z I_z} \left[ \cos(\phi_z + \psi_z) \right] \left\{ x \frac{\delta p}{p} + \sqrt{2 \beta_x I_x} \left[ \cos(\phi_x + \psi_x) \right] \right\}
\]

involves both \( x \) and \( z \), and couples the two motions together. The first term gives a horizontal field, and hence a vertical orbit error, which depends on momentum or position of the equilibrium orbit. This is usually called "vertical dispersion", or "median plane tilt" and is usually accompanied by coupling. The effect on the orbit depends on dispersion \( \eta \) and can be calculated by the same methods used above.

The coupling term becomes

\[
A_{skew} = B' (\theta) xz = 2B' (\theta) \sqrt{2 \beta_z I_z} \left[ \cos(\phi_z + \psi_z) \right] \left[ x \frac{\delta p}{p} + \sqrt{2 \beta_x I_x} \left[ \cos(\phi_x + \psi_x) \right] \right] \]

The two terms have different dependences and so must be treated differently. The first will be slowly varying when \( \nu_x - \nu_z \approx \text{integer} \), while the second will be slowly varying if \( \nu_x + \nu_z \approx \text{integer} \). These conditions are called difference and sum resonance respectively. In a similar way to above, we Fourier expand the appropriate term for difference resonance.

\[
B' \sqrt{\beta_x \beta_z} e^{i(\psi_z)} = \sum_{-\infty}^{+\infty} h_k e^{\pm i(\theta_k + \psi_k)}
\]

Ignoring the other term, the Hamiltonian becomes
\[ K = I_x v_x + I_z v_z + \sqrt{I_x I_z} \sum_{k=-\infty}^{+\infty} \beta_k \cos(\phi_x - \phi_z + k\theta + \delta_k) \] 9.32

We ignore all terms except the slowly varying one for some \( p = k \) where \( v_x - v_z + p \approx 0 \) (Note that \( p \) might be positive or negative). Then again we transform away the \( \theta \) dependence by

\[ F_z = I_1 (\phi_x - \phi_z + p\theta + \delta_p) + I_2 \phi_z \] 9.33

This leads to

\[
\begin{align*}
I_x &= I_1, I_z = I_2 - I_1, \\
\phi_1 &= \phi_x - \phi_z + p\theta + \delta_p, \quad \phi_2 = \phi_z \\
K' &= I_1 (v_x - v_z + p) + I_2 v_x + h_p \sqrt{I_1 (I_2 - I_1)} \cos \phi_1
\end{align*}
\] 9.34

\( K' \) is independent of \( \phi_2 \) and \( \theta \), so both \( K' \) and \( I_2 \) are constants of the motion. Note that since \( I_2 \) is the sum \( I_x + I_z \), the motion is bounded no matter what the initial conditions. "Energy" (I) is pumped back and forth between the x and z motions, the sum remaining constant. Only at the resonance can energy be exchanged completely between x and z motion. The frequency at which the energy is exchanged is just the difference frequency \( \epsilon = v_x - v_z + p \) of the coupled motion from the resonance. This turns out to be a general result for resonances of all order.

In the case of sum resonance, we proceed in a similar way. We Fourier expand

\[ B' = \beta_x \beta_z e^{\pm i(\psi_x + \psi_z)} = \sum_{k=-\infty}^{+\infty} \beta_k e^{\pm i(k\theta + \delta_k)} \] 9.36

and the Hamiltonian becomes

\[ K = I_x v_x + I_z v_z + \sqrt{I_x I_z} \sum_{k=-\infty}^{+\infty} \beta_k \cos(\phi_x + \phi_z + k\theta + \delta_k) \] 9.37

We select only the stationary term where \( \epsilon = v_x + v_z + q \approx 0 \) for some \( q \). We transform by

\[ F_2 = I_1 (\phi_x + \phi_z + q\theta + \delta_q) + I_2 \phi_z \] 9.38

so that

\[
\begin{align*}
I_x &= I_1, I_z = I_2 + I_1, \quad \text{or} \quad I_2 = I_x - I_z, \quad I_1 = I_x \\
\phi_1 &= \phi_x + \phi_z + q\theta + \delta_q, \quad \phi_2 = \phi_z \\
K' &= I_1 (v_x + v_z + p) + h_p \sqrt{I_1 (I_x + I_z)} \cos \phi_1
\end{align*}
\] 9.39

Again, \( K' \) and \( I_2 \) are constant, but now \( I_2 \) is the difference of the x and z "energies" instead of the sum. Then the motion is not automatically bounded as at a difference resonance. To investigate the condition of stability, we look for separatrices where \( \frac{d\phi_1}{d\theta} = 0 \) as we did for half integral resonance. Unless these occur at \( \phi_1 = \pm \pi \) or 0, the motion has a non-zero value of \( \frac{dI_1}{d\theta} \) and so the motion "hangs up" on the separatrix. Because of the dependence of the last term on \( I_1 \), we look for values \( I_1 = \alpha I_2 \) which satisfy \( \frac{d\phi_1}{d\theta} = 0 \). Now
\[ \frac{d\phi_1}{d\theta} = \epsilon + \frac{h_q}{2} \frac{I_2 + 2I_1}{\sqrt{I_1(I_2 + I_1)}} \]
\[ = \epsilon + \frac{h_q}{2} \frac{1 + 2\alpha}{\sqrt{\alpha(1 + \alpha)}} \cos \phi_1 = 0, \text{ or } \cos \phi_1 = -2D \frac{\sqrt{\alpha(1 + \alpha)}}{1 + 2\alpha} \]

where \( D = \frac{\epsilon}{h_q} = \frac{v_x + v_z + p}{h_q} \). If values for \( \alpha \) can be found where \( |\cos \phi_1| \leq 1 \), then the motion is unstable. The boundary of stability is where \( |\cos \phi_1| \leq 1 \). With some manipulation, this becomes
\[ (1 + 2\alpha)^2(1 - D^2) \geq -D^2 \]
So long as \( D \geq 1 \) it is possible to find a real value of \( \alpha \) to satisfy the previous formula. Then the condition for instability is \( h_q \geq \epsilon \). We can call \( h_q \) the "width" of the stopband or resonance. Thus the sum resonance shows great similarity to the half integral resonance. Outside the stopband, in-phase beating in \( x \) and \( z \) motion occurs at the difference frequency from the resonance \( \epsilon \).

Sextupole Errors

We now consider the next multipole, \( n = 3 \), the sextupole. First we consider the systematic effects due to off momentum orbit motion. The sextupole components are given by
\[ A_{3n} = \frac{B''(\theta)}{6} \left[ x^3 - 3xz^2 \right] \quad A_{3s} = \frac{B''(\theta)}{6} \left[ 3x^2z - z^3 \right] \]

If we let \( x = X + x_p \frac{\Delta P}{P} \), we find terms proportional to \( \eta \frac{\Delta P}{P} \),
\[ A_{3n} = \frac{B''(\theta)}{6} \left[ X^3 - 3XZ^2 + 3x \frac{\Delta P}{P} (X^2 - Z^2) \right] + \ldots \]
\[ A_{3s} = \frac{B''(\theta)}{6} \left[ 3X^2Z - Z^3 + X \frac{\Delta P}{P} XZ \right] + \ldots \]

From the point of view of the transverse motion, there appears a normal quadrupole which is dependent on momentum (or closed orbit error). This causes momentum dependent tune shifts, or chromaticity. The amounts can be calculated using the methods developed above. Sextupoles are used to correct the natural chromaticity of rings. The tune shifts in \( x \) and \( z \) are proportional to \( \beta_x \eta \) and \( \beta_z \eta \) respectively. By locating independent sextupoles at maximum \( \beta_x \) and maximum \( \beta_z \) the chromaticity of both \( x \) and \( z \) can be controlled. In fact more complicated strategies are usually employed to control the effects of resonances. In the skew component there appears a skew quadrupole which is proportional to momentum. This causes the \( x-z \) coupling to depend on momentum, which can frustrate attempts to correct coupling by the introduction of skew quadrupoles.

If we consider the phases of the possible combinations of \( \phi_x, \phi_z, \) and \( \theta \) in the normal and skew terms above we see that conditions of resonance (slow variation of phase) can occur in the following cases: (\( n \) is different in each case, in general)
\[ \nu_x = n: \text{ Normal sextupole} \]
\[ 3\nu_x = n: \text{Normal sextupole} \]
\[ 2\nu_x \pm \nu_z = n: \text{Skew Sextupole} \]
\[ \nu_x \pm 2\nu_z = n: \text{Normal sextupole} \]
\[ 3\nu_z = n: \text{Skew Sextupole} \]
\[ \nu_z = n: \text{Skew Sextupole} \]

We see that the integer resonance is also a nonlinear resonance. We recall that integral resonances are half integral also, being driven by quadrupole errors. In fact, each time a higher order term is introduced, lower order resonances are driven in a new way. This is often called "feeddown". We might suspect that the sum resonances will be more dangerous than the difference resonances in analogy to the linear coupling case.

The simplest case of nonlinear resonance is the one dimensional resonance \( 3\nu_x = n \). This resonance is used for extracting the proton beam from some synchrotrons and merits attention. The resonance is driven by the \( x^3 \) term in the sextupole field above. The term in the Hamiltonian is

\[
A_{3n} = \frac{RB''(\theta)}{6B\beta} \[2I_x \beta_x \]^3/2 \cos^3(\phi_x + \psi_x) = \frac{RB''(\theta)}{6B\beta} \[2I_x \beta_x \]^3/2 \cos^3(\phi_x + \psi_x) + 3\cos(\phi_x + \psi_x)
\]

The first term is the driving term for the \( 3\nu_x = n \) resonance, while the second is the driving term for the \( \nu_x = n \) "feeddown" resonance. We expand

\[
\frac{RB''(\theta)}{24B\beta} \[2\beta_x \]^3/2 \sum_{m=-\infty}^{\infty} d_m e^{i(m\theta + \lambda_m)}
\]

Then the Hamiltonian becomes

\[ K = I_x \nu_x + (I_x)^{3/2} \sum_{m=-\infty}^{\infty} d_m \cos(3\phi_x + m\theta + \lambda_m) \]

We keep only the term \( m = -n \), where \( 3\nu_x \approx n \). We make a canonical transformation

\[ F_2 = J(3\phi_x - n\theta + \lambda_n) \]

\[ \therefore I_x = 3J \]

so

\[ &\gamma = (3\phi_x - n\theta + \lambda_n) \]

\[ K' = J\varepsilon + D(3J)^{3/2} \cos 3\gamma, \text{ where: } \varepsilon = (\nu_x - n/3) \]

There are three fixed points, for \( \gamma' = 0 \) at \( \gamma = 0, \pm \pi/3, \pm 2\pi/3, \pi \). \( J' = 0 \) (if \( D/\varepsilon < 0 \)) at \( \gamma_f = 0, \pm 2\pi/3 \), and at \( J_{f} = \frac{3}{243}(\varepsilon/D)^2 \)
Octupole Errors

A different form of systematic effect is caused by octupole terms, similar to those investigated for the pendulum earlier. The quartic terms cause the tunes $\nu_x$ and $\nu_z$ to depend on amplitude. Here, since the motion has more degrees of freedom, the effects are more complicated. The normal octupole leads to a term in the Hamiltonian

$$\frac{RB'''}{4!B\rho} \left[ x^4 - 6x^2z^2 + z^4 \right]$$

We substitute angle action variables in each term, so that we have $\cos^4$ and $\cos^2\cos^2$ terms. We expand the cosines in exponentials and consider only those terms which are independent of the $\gamma$'s. We find for them;
Then when we find \( \gamma'_x \) and \( \gamma'_z \) we find another term, so that
\[
\gamma'_x = \frac{\partial K}{\partial I_x} = \nu'_x + \frac{RB''}{8B} \beta'_x I_x - 2\beta'_x \beta_x I_x z + \beta'_x I_x z + \beta_x z^2 I_x z^2
\]

Thus the normal octupole causes each tune to depend on its own amplitude and the other amplitude \( I \).

General Resonance

Now let us consider a general term in the Hamiltonian
\[
\pm B N^{-1} \left( \frac{1}{N-1} \right)^{N} C_N \left( \sum_{p+q=N} p^p q^q \right) \quad (p+q=N), \quad \pm B N^{-1} \left( \frac{1}{N-1} \right)^{N} C_N = Q(\theta)
\]

We substitute for \( z \) and \( z \) in terms of the \( I \)'s and \( \phi \)'s, and express the cosine terms in exponentials. The highest order terms will be of the form
\[
\frac{Q(\theta)}{2^{p+q}} \left( \sum_{p+q=N} p^p q^q \right) \exp \pm i[p(\phi_x + \psi_x) + q(\phi_z + \psi_z)]
\]

Fourier expand
\[
\frac{Q(\theta)}{2^{p+q}} \left( \sum_{p+q=N} p^p q^q \right) \exp \pm i[p(\phi_x + \psi_x) + q(\phi_z + \psi_z)] = \frac{1}{2} \sum_{-\infty}^{\infty} \frac{d_k}{k} e^{i(k\theta + \xi_k)}
\]

Then the term in the Hamiltonian becomes
\[
(\frac{I_x}{2})^2 (\frac{I_z}{2})^2 \sum_{k=-\infty}^{\infty} d_k \cos(p\phi_x + q\phi_z + k\theta + \xi_k)
\]

Again we look for a term where the phase is slowly varying, to wit
\[
p \nu'_x + q \nu'_z + r = 0
\]

We transform to rotating coordinates by
\[
F_2 = I_1(p \phi_x + q \phi_z + r \theta + \xi_r), \quad I_x = p I_1 + I_2, \quad \phi_1 = p \phi_x + q \phi_z + r \theta + \xi_r, \quad \phi_2 = q \phi_z
\]

\[
K' = I_1(p \nu_x + q \nu_z + r)
\]
Now both $K'$ and $I_z = \frac{I_z}{q} - \frac{I_x}{p}$ are constant since $K'$ is independent of $\theta$ and $\phi_2$. Then if $p$ and $q$ have the same sign, (sum resonance) $I_z$ and $I_x$ can both grow, just as for the quadrupole-driven difference above. We can search for fixed points

\[
I_1' = d_r[pI_1]^2[p(I_1 + I_2)]^2 \sin \phi_1 = 0 \implies \phi_1 = 0 \text{ or } \pi \tag{9.55}
\]

\[
\phi_1' = p\nu_x + q\nu_z + r + d_r\frac{pq}{2}[pI_1]^2[q(I_1 + I_2)]^2 \left[ (q + p)I_1 + pI_2 \right] \cos \phi_1 \tag{9.56}
\]

If the motion is exactly on resonance, i.e. $p\nu_x + q\nu_z + r = 0$, the only solution to $\phi_1' = 0$ is $I_1 = 0$. For non-zero values of $p\nu_x + q\nu_z + r$, $\phi_1' = 0$ can be solved for values $I_{fp}$ at either $\phi_1 = 0$ or $\phi_1 = \pi$. For values of $I_1$ larger than $I_{fp}$, we can find values of $\phi_1 = \pm \phi_s$ where $\phi_1' = 0$. The locus of these points comprises the outward two branches of the separatrix.

### Slow Extraction

As an application of these methods, consider “slow” extraction from a synchrotron. The goal is to create a stable region in the phase space of one (usually horizontal) dimension outside of which the amplitude grows radically so that most particles can “jump” across a septum into a channel which removes them from the accelerator.

Particle motion can be described by an approximate Hamiltonian function $K$ given by

\[
K' = I_x\left( \nu_x + \delta \nu, \frac{r}{2} + q \cos 2\gamma, + \frac{kI^2}{2} \right) \tag{9.57}
\]

Where the first part of $K'$ is 9.25, and $k$ comes from equation 9.50

\[
k = \frac{\partial \nu_x}{\partial I_x} = \left( \frac{RB \beta^2}{8Bz} \right) \tag{9.58}
\]

The equations of motion are

\[
I_x' = 2I_x \delta \nu \sin 2\gamma, \text{ and } \gamma_x' = \nu_x + \delta \nu_x - \frac{r}{2} + q \rho \cos 2\gamma + kI_x = \epsilon + \Delta \nu \cos 2\gamma + kI_x \tag{9.59}
\]

Here $\epsilon$ is the distance in tune from the resonance, and $\Delta \nu$ is half the stopband width of the resonance. Fixed points occur when

\[
\gamma_x = 0, \pm \frac{\pi}{2}, \pm \pi, etc. \text{ and } I_x = -\frac{\epsilon \pm \Delta \nu}{k} \tag{9.60}
\]
The inner fixed points corresponding to the separatrix containing the smallest area are, for this case
where $\varepsilon$, $\delta\nu$, and $-k$ are positive, for $\gamma = \pm \pi/2$
\[ I_{si} = \frac{\Delta\nu - \varepsilon}{k} \]  
Outer fixed points at $\gamma = 0$ and $\pm \pi$ occur at larger $I$ at
\[ I_{so} = \frac{-\Delta\nu - \varepsilon}{k} \]
$I_{s,i}$ goes to zero and $I_{s,o}$ goes to $2\varepsilon/k$ as $\Delta\nu$ goes to $\varepsilon$, so the size of the stable region can be controlled by
changing $\Delta\nu$. Alternatively, one could keep $\Delta\nu$ constant and change $\delta\nu$ to shrink the stable region. As
the area of the stable region shrinks past the area of the beam, particles “leak out” of the region at the
unstable fixed points and proceed outward, along the separatrix, growing rapidly in amplitude.
However, because the tune shifts with amplitude, the particles fall out of phase with the resonance and
return (around the outer fixed point) to the neighborhood of the inner fixed point. Thus one has a
limited region over which the amplitude can grow to cross a septum. The main features of these ideas
are illustrated in the figure below. The design values for the extraction system yield ($\nu_0 = 0.6$)
\[ k = 151.83 \text{m}^{-1} \]
\[ \Delta\nu = 0.04526 \]
\[ \delta\nu = -0.05476 \]
\[ \varepsilon = 0.05474 \]
\[ I_s = 13.8 \mu\text{m} \]
Figure 1. Extraction orbits in the neighborhood of the separatrix in the LLMA. Two orbits are plotted each revolution, one just inside, and one starting just outside the separatrix.
Injection

Matching -

The beam must be transported from its source to the accelerator by a beam transport line. In order to avoid emittance dilution, the beam must be matched. That is, the $\alpha$ $\beta$ $\gamma$ $\eta$ characterizing matched ellipses and superimposition must be equal to those in the accelerator. This is usually the most difficult part of injection.

Single Turn Injection

90° in helix phase

$\beta = \frac{x_1}{\sqrt{\beta_0'^2 \sin^2 \gamma}}$

Example: $x_1 = 5$ cm, $\beta = \beta_0 = 10$ cm, $\gamma = \frac{\pi}{2}$

$\beta \delta$ of kicker $= 0.1$ Tm = 100 g.m.

Drive with so-called

Drive with so-called

For example, if $w = 15$ cm, $g = 5$ cm, $l = 0.5$ m,

$L = \frac{\mu_0 w \delta}{\delta}$

$L/R$ time $= L/2\mu_0 = 38$ nsec., or rise time is 83 nsec.

$I = \frac{B_0}{\mu_0} \frac{0.02 \times 0.05}{9 \times 10^{-3}} = 500$ A.
The amplitude of the current pulse is 40 kV (= 500 \times 80)

Storage Cable

Kicker Magnet

**SEP IUM.**

The field should be uniform so the conductor must be in a gap.

[Copper septum]

Insulator

Laminated Fe \( \approx 0.1 \) mm thick.

If \( B = 0.3T \) and \( Bp = 2 \) Tm, \( P \approx 7 \)

Such septa have been made (pulsed) for 1 T fields

\[ L \quad \text{and} \quad I = 20 \text{kA} \]

\[ C = 500 \text{J/m} \]
Charge Exchange Injection. ($N^-$)

The beam is passed through a thin ($\sim 100\,\mu m$) foil to strip the electrons.

Typically, for 200 MeV $N^-$, the foil is 1000 µg/cm$^2$ of pyrolytic graphite. The bump magnets are pushed on and off to remove the circulating beam from the foil. This minimizes the damage to the foil, and the multiple scattering of the protons.
Extraction

**Single Turn**. The inversion of single-turn injection. Usually, the momentum is greater so the magnets are more difficult. The kicker is chosen, and multipolar coils are used to increase the current, while maintaining the same rise time. Multiple kicks are then needed also to get the required deflection. This can be more difficult, and sometimes a Lambertson is used (see below).

**Multiple Turn**

One may resonate to help extract the beam. In principle, any order resonance can be used. I will describe half-integer resonance extraction with Octupoles to control the tune. The acceleration tune is shifted toward a resonance, $\nu = \nu_0/2$ (an integer), by quadrupoles which have an average component and a harmonic component at period of $\nu$. The tune changes with amplitudes being closest to the resonance at larger amplitudes. This is achieved with a mean value of Octupoles.

An approximate Hamiltonian is:

$$K = \varepsilon I + \frac{k}{2} I^2 + IS_x \cos 2\sigma$$

$$\varepsilon = \nu_0 + \alpha \nu - \frac{\nu}{2}$$

$$I = \frac{1}{2} \left( \beta x^2 + 2\alpha x x' + \beta x' x'' \right)$$

$$\sigma = M - \frac{\beta \theta}{2}$$

$$\tan \mu = x + \frac{\beta x'}{x}$$
\[ \alpha \nu = \frac{1}{\sqrt{y^2}} \left( \int \frac{d\alpha \beta x B^1}{\beta \epsilon} \right) = \sum \beta \beta' \delta \epsilon \]
\[ \delta \nu = \frac{1}{\sqrt{y^2}} \left( \int \frac{d\alpha \beta x B^1 \cos \theta}{\beta \epsilon} \right) = \sum \frac{\beta}{\sqrt{y^2}} \frac{B' \epsilon \cos \theta}{\beta \epsilon} \]
\[ \kappa = \frac{1}{16 \pi} \left( \int \frac{d\alpha \beta x B''^1}{\beta \epsilon} \right) = \sum \frac{\beta^2}{\alpha} \frac{B'' \epsilon \cos \theta}{\beta \epsilon} \]

The fixed particle are at 0 or \( \pi \)

and \( I = \frac{\delta - \xi}{\kappa} \) (size of stable region)

The beam can be extracted by gradually shifting \( \delta \nu \) toward the resonance, as the separatrix shrinks. Alternatively, chromaticity can be added; thus accelerating the beam.

\[ P \]
\[ P_1 \]
\[ P_2 \]

moves the tune toward the resonance. The acceleration can be by keeping the beam bundled with an RF system, or stochastically. As an alternative, the transverse can be stochastically heated, and the rest of the system is stationary.

Electrostatic Septa.

A very thin septum is needed. The best has been using wire meshes in an electrostatic septum.

[Diagram of an electrostatic septum with a beam passing through it and a voltage of 60 kV/cm]
Lambertson Septum Magnet (DC type)

Instead of having the septum magnet deflect in the same plane as the electrostatic septum, better performance can be obtained by deflecting in the other plane.

Then a 1 T change in field can be gotten across a 1 mm septum.

![Diagram](image)

Circulating beam channel (deflected up or down)

The Lambertson can be run DC and has only 1.5 mW where the beam loss occurs. They are extremely rugged.
A Pulsed Septum Magnet for the APS*

L. R. Turner, D. G. McGhee, F. E. Mills, and S. Reeves
Advanced Photon Source, Argonne National Laboratory
9700 South Cass Avenue, Argonne IL 60439, U. S. A.

Abstract
A pulsed septum magnet has been designed and constructed for beam injection and extraction in the Advanced Photon Source at Argonne National Laboratory. The magnets will be similar for the Positron Accumulator Ring (PAR), the Injector Synchrotron, and the Storage Ring. The septum itself is 2 mm thick and consists of 1-mm-thick copper and S1010 steel explosion-bonded together. The PAR magnet is driven by a 1500-Hz, 12-kA half sine wave current pulse. The core is made of 0.36-mm-thick laminations of silicon steel. The nearly uniform interior field is 0.75 T and the exterior field is 0.0004 T at the undisturbed beam position and 0.0014 T at the bumped beam position. Testing of the magnet awaits completion of the power supply.

I. INTRODUCTION
Pulsed septum magnets are used for injection and extraction of the particle beam in circular accelerators. For the Advanced Photon Source (APS), now under construction at Argonne National Laboratory (ANL), these magnets will be used in the 450-MeV Positron Accumulator Ring (PAR), the 450-MeV to 7-GeV Injector Synchrotron, and the 7-GeV Storage Ring. Requirements for the septum magnets include good field homogeneity in the aperture, low stray fields, a thin septum, and rapid excitation and discharge. Both transformer-driven and direct-driven septum magnets were analyzed. Analysis showed that for a septum consisting of 1-mm-thick copper and S1010 steel strips explosively bonded together, the transformer-driven magnet has lower external fields and more uniform interior fields than the direct driven, if the septum current is constrained to all flow in the gap region.

The design of the septum magnet described here grows out of earlier septum magnets at ANL [1] and Fermilab [2]. Construction of the first magnet is complete, and testing awaits completion of the power supply. Table 1 shows the parameters of the septum magnet.

II. COIL AND CORE ASSEMBLY
The coil assembly with septum and coil is shown in Figure 1, and the cross section of the septum magnet is shown in Figure 2. The single turn primary coil consists of a continuous length of OFHC copper of rectangular cross section, 15.9 mm by 6.35 mm, with a 4.8-mm diameter hole for water cooling. The coil is formed to the final shape in a winding fixture. The lead ends are left long enough to extend beyond the vacuum enclosure, ensuring there are no braze joints within the vacuum system.

Table 1

<table>
<thead>
<tr>
<th>Septum Magnet Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Field</td>
<td>0.75 T</td>
</tr>
<tr>
<td>Magnet Length</td>
<td>400 mm</td>
</tr>
<tr>
<td>Aperture Width</td>
<td>70 mm</td>
</tr>
<tr>
<td>Aperture Height</td>
<td>20 mm</td>
</tr>
<tr>
<td>Septum Thickness</td>
<td></td>
</tr>
<tr>
<td>copper</td>
<td>1.0 mm</td>
</tr>
<tr>
<td>steel</td>
<td>1.0 mm</td>
</tr>
<tr>
<td>Pulsed Current drive</td>
<td>5960 A</td>
</tr>
<tr>
<td>septum</td>
<td>5954 A</td>
</tr>
<tr>
<td>Pulse Shape</td>
<td>half sine wave</td>
</tr>
<tr>
<td>Pulse Frequency</td>
<td>1500 Hz</td>
</tr>
</tbody>
</table>

Figure 1. The core assembly with septum and coil, ready for installation in the vacuum enclosure.

Because the coil is inside the vacuum system, polyamide was chosen as insulating material for its good vacuum and dielectric properties. It has good mechanical properties both at operating temperatures and at 260 °C during vacuum bake-out. Insulating sleeve parts are machined from the polyamide and then cleaned by total immersion in an ultrasonic bath. The coil is then enclosed by the sleeve parts, and the laminations are stacked around this assembly in the core stacking fixture to form the core.

The core is made of 0.36-mm-thick laminations stamped from AISI type M-22 steel, with an AISI C-5 surface insulation coating. The laminations are baked at 750 °C in a vacuum oven at a pressure of 1x 10^-4 Torr, a cleaning process for ultra-high vacuum systems. The laminations are stacked with 13-mm-thick low-carbon steel end plates in the stacking fixture, which maintains the required surface tolerance. Two 6-mm-thick low-carbon-steel bars are welded to the lamination stack and end plates to constrain the core. The core assembly is structurally robust and can easily withstand the electromagnetic forces it encounters.

III. SECONDARY TURN AND VACUUM ENCLOSURE

The septum is a crucial part in the design of the magnet. Because it is thin and subject to magnetic, thermal, and machining forces, the proper bonding between copper and steel is essential. Explosion bonding was selected over brazing or roll bonding because it can consistently achieve void-free, very strong mechanical bonds. After 3.2-mm-thick low carbon steel and C102 OFE copper are explosively bonded together, the septum is machined to its final thickness: 1 mm of copper and 1 mm of 1010 steel.

The weld between the septum and the core must carry the combined thermal and magnetic forces. Electron-beam welding was selected for this critical joint because of the dimensional precision that can be attained and because of the depth of penetration of the weld. The resulting welds at the transition between the laminations and the septum steel are very smooth and continuous. Examination of test samples showed that these joints were not contaminated by copper.

The completed core assembly is placed inside the vacuum enclosure, made of plates of the same bonded steel and copper as the septum. The secondary turn is made up of the copper panels that face the inside of the enclosure. The enclosure is welded, then cleaned by total immersion in an ultrasonic bath. The secondary turn is completed by welding the septum to the copper inner face of the enclosure. After electrical checks are completed, the top of the enclosure is welded in place. Vacuum feedthroughs and flanges complete the assembly.

IV. ELECTROMAGNETIC DESIGN

The electromagnetic design involved choosing between a directly driven or transformer-driven magnet, choosing the form of the septum, and considering the consequences of current above and below the level of the aperture. The electromagnetic design was carried out with the magnetostatic, steady-state AC and transient solvers of the 2-D analysis code OPERA-2D (formerly P97D) [3].

Both directly driven and transformer-driven septum magnets were considered. For the direct drive, the current in the septum of necessity is exactly that which is needed to produce the required field in the gap. For the transformer drive, the driven current determines the field in the gap, and the septum current is somewhat less, just enough to establish a zero current in the exterior region. The transformer driven case was found to yield an exterior field about a factor of ten lower than the directly driven case, at both the bumped beam position and the undisturbed beam position as seen in Fig. 3.

For the composite septum, the steel part of the septum contains the flux that penetrates the copper part. That effect was found to more than compensate for the fewer skin depths of copper with the composite septum.

It is necessary to locate the septum in slots in the yoke, in order to restrain it against magnetic forces. Consequently, there is some current flow above and below the 20-mm-thick region of the aperture and septum. Some earlier experiences [2] suggested that these currents could lower the homogeneity of the field in the aperture and increase the field leakage outside. But analysis of geometries with and without such currents showed effectively no differences. Uniformity of the field in the aperture is shown in Figure 4.
V. SEPTUM POWER SUPPLY

The four single turn transformer septum magnets for the PAR, the Injector Synchrotron, and the Storage Ring are powered by capacitor discharge circuits. These are designed to produce half-sine-wave pulses with a base width of 1/3 ms and peak currents repeatable within 0.05% and adjustable from 470 A to 4.7 kA. The capacitor discharge circuits are transformer coupled for impedance matching to the magnet. The peak currents of the transformer secondaries range from 11.4 kA to 16.888 kA. Figure 5 shows a diagram of the power supply. The switch S4 may either be gated or not, depending on the need to reset the magnet steel.

VI. CONCLUSIONS

This design is found to satisfy the conditions required of a septum magnet, both mechanically and electromagnetically. Voltage and inductance testing has begun, and field properties will be measured when the power supply is available.

VII. REFERENCES


[3] OPERA-2D (PE2D) is available from Vector Fields, Inc. Aurora, IL, USA.