Lecture 9: Synchrotron Radiation
Summary

A. Radiated power
B. Adiabatic invariants, phase space damping
C. Beam size
D. Wigglers
Synchrotron Radiation

Synchrotron Radiation is emitted by particles which are accelerated. Since the radiated power (for a given accelerator geometry and magnetic lattice) is inversely proportional to $m^4$, it can usually be ignored for any particles except electrons and positrons. So far, no accelerator of ions has been constructed in which synchrotron radiation plays an important role. However, this will change in the next generation of hadron colliders. A useful image of the process is to imagine an electron gyrating in a magnetic field, creating an oscillating electric dipole moment. The electron radiates at the gyrofrequency $eB/mc$, the radiation polarized in the plane of oscillation. Now consider a relativistic electron circulating in a magnetic field $B$. In the electron's rest system the field is Lorentz boosted to $\gamma B$ and the electron radiates at $\gamma eB/mc$. Transforming back to the laboratory, the photon frequency is transformed by another factor of $\gamma$ to about $\gamma^2 eB/mc$ and the dipole radiation pattern is Lorentz transformed into a cone of angle $1/\gamma$. 
The total power radiated can be calculated classically by calculating the dipole power and transforming it into the laboratory frame. The actual emission of energy is by the emission photons of energy \(\varepsilon\), say, a fact which profoundly affects the orbits of the electrons. The spectrum, the mean number of photons of energy \(\varepsilon\) emitted per unit time, \(\dot{N}(\varepsilon)\), has been calculated, but we will only need to know the mean energy lost per unit time \(\langle \dot{N}\varepsilon \rangle\), and the mean energy squared lost per unit time \(\langle \dot{N}\varepsilon^2 \rangle\).

The radiated power is given by

\[
P_\gamma = \langle \dot{N}\varepsilon \rangle = \frac{2e^2}{3c} \left( \frac{\gamma^2}{\beta^2} + \frac{\beta^4 c^2 \gamma^4}{\rho^2} \right) = \frac{2mc^2 r_0}{3c} \left( \frac{\gamma^2}{\beta^2} + \frac{\beta^4 c^2 \gamma^4}{\rho^2} \right)
\]
where we have included the acceleration in the longitudinal direction as well as the transverse due to the orbit curvature in the magnetic field. In all practical cases of interest, where $E_{el}$ is the accelerating electric field, the longitudinal term is negligible, of order \( \left( \frac{E_{el}}{\beta \gamma c B} \right)^2 \) \( (E_{el} \text{ in V/m and } B \text{ in T}) \) compared with the transverse term. We will further set $\beta = 1$ and write

\[
8.2 \quad P_\gamma = \langle N \epsilon \rangle = D E^2 B^2 \quad \text{where} \quad D = \frac{2r_e^2 c^3}{3(m c^2)^2} \quad \text{and} \quad E \text{ is particle energy.}
\]

D is about 2.4 kHz/GeV T$^2$. The angular distribution is characterized by (spatial angle)

\[
8.3 \quad \langle \psi^2 \rangle = \frac{1}{\gamma^2}
\]

so that the electron receives a random transverse momentum kick of magnitude \( \frac{\epsilon \psi}{c} \). This kick is very small compared to the changes made by shifts in reference orbits, and is only detected in the vertical motion in case there is small coupling.
The spectrum is characterized by the "critical energy" $\varepsilon_c$, given by

$$\varepsilon_c = \frac{3}{2} \frac{\hbar c}{\rho} \gamma^3$$

The mean square photon energy radiated per unit time is given by

$$\langle \dot{\mathcal{N}}\varepsilon^2 \rangle = \frac{55}{2^3 3^{3/2}} \mathcal{P}_\gamma \varepsilon_c$$

We will sometimes use the photon momentum $\delta P = \varepsilon/c$, and the number of photons per unit s

$$N' = \dot{N} \frac{1 + \Omega \left( x + x_p \frac{\Delta E}{E} \right)}{c}$$

from $dN/ds = (dN/dt)(dt/d\sigma)(d\sigma/ds)$.
Let us calculate the changes in the adiabatic invariants $J_x = PW_x$, $J_z = PW_z$, and $W_E$ where

$$W_E = \frac{(\Delta E)^2}{Y} + Y(\Delta t)^2$$

is the invariant for the phase oscillations with matching parameter $Y$. (Note that we have used time and energy rather than the phase $\tau$ and the variable $W$ used in 7.21. $Y$ is the ratio $\frac{\Delta E_{\text{max}}}{\Delta t_{\text{max}}}$ for the phase oscillation and is numerically equal to the quantity $Z$ of Eq. 7.21.) We will ignore the transverse momentum kick of the photon for now. Then the angles in space $X'$ and $Z'$ do not change, but since the momentum changes, the betatron angle and amplitude change, so the changes include

$$\delta x = +x_p \frac{\delta P}{P}$$

$$\delta x' = +x'_p \frac{\delta P}{P}$$

$$P \rightarrow P - \delta P$$
First let us calculate the changes in $J_z$. Since $z$ and $z'$ do not change, $W_z$ does not change, and the only change in $J_z$ is through the change in $P$ according to Eq. 8.6.

Adding up all photon energies,

\[
\delta J_z = -W_z \delta P = -J_z \frac{DE^2B^2}{Pc^2} \left(1 + \Omega \left(x + x_p \frac{\Delta E}{E}\right)\right) \delta s \quad \text{*see notes}
\]

Evaluating at $x = z = \Delta E = 0$ and averaging over the circumference, 

\[
\frac{\dot{J}}{J} = -DE\left\langle B^2 \right\rangle \equiv -\tau_z
\]
\[ \delta J_z = -W_z \delta P = -J_z \frac{DE^2B^2}{Pc^2} \left(1 + \Omega \left(x + x_p \frac{\Delta E}{E}\right)\right) \delta s \]

This is from

\[ \delta J_z = -W_z \delta P = -\frac{J_z}{P} \delta P = -\frac{J_z}{P} \frac{\delta \varepsilon}{c} = \frac{-J_z N'}{Pc} \delta s \]
Next let us calculate the changes in $J_x$. Now both $x$ and $x'$ change. Further the energy loss depends on $x$ through its dependence on arclength and the magnetic field variation. There are terms linear and quadratic in $x$ and $x'$ and terms quadratic in $x_p$ and $x'_p$.

$$\delta J_x = -W_x \delta P$$

8.11  
$$+P \left[ (2\alpha x' + 2\gamma x)x_p \frac{\delta P}{P} + (2\alpha x + 2\beta x')x_p' \frac{\delta P}{P} \right] \quad \Leftarrow \text{middle terms}$$

$$+P\left[ \gamma x_p^2 + 2\alpha x_p x'_p + \beta x'_p^2 \right] \left( \frac{\delta P}{P} \right)^2$$

*see notes

We next sum up all photon energies. The first term is the same as for $z$. The second and third terms do not average to zero because the energy loss depends on $x$. If we use the phase-amplitude forms 4.33 for $x$ and $x'$, we can average over phase to find that

8.12  
$$2\langle x^2 \rangle = \beta W, \quad 2\langle xx' \rangle = -\alpha W, \quad 2\langle x'^2 \rangle = \gamma W$$

**see notes
\[ W = \gamma x^2 + 2\alpha xx' + \beta x'^2 \]
\[ \frac{\partial W}{\partial x} = 2\gamma x + 2\alpha x' \]
\[ \frac{\partial W}{\partial x'} = 2\alpha x + 2\beta x'^2 \]

\[ \delta J_x = -W_x \delta P + P \left[ \frac{\partial W}{\partial x} \delta x + \frac{\partial W}{\partial x'} \delta x' \right] \frac{\delta P}{P} + P \left[ W(x_p) \right] \frac{\delta P}{P} \left( \frac{\delta P}{P} \right) \]

**

Use

\[ x = \sqrt{\beta W} \cos(\psi + \delta) \]

\[ x' = \sqrt{\frac{W}{\beta}} \sin(\psi + \delta) - \alpha \cos(\psi + \delta) \]

(eqns 4.33)

then take the averages.
We Taylor expand the mean momentum loss per meter using Eq. 8.2 and 8.6, perform the averages over betatron phases and the lattice, and find for the middle terms

\[
8.13 \quad \left[-\alpha^2 W + \beta \gamma W\right] \text{DE} \left\langle (2B'B' + B^2 \Omega)x_p^{'}\right\rangle \delta s + \left[\alpha \beta W - \beta \alpha W\right] \text{DE} \left\langle (2B'B' + B^2 \Omega)x_p^{''}\right\rangle \delta s
\]

\[
= W \text{DE} \left\langle (2B'B' + B^2 \Omega)x_p^{'}\right\rangle \delta s
\]

The last term in 8.11 is positive definite, does not depend on \( W \) and can be considered a diffusion or heating term. It depends on the mean square energy loss \( \left\langle \dot{\gamma} \varepsilon^2 \right\rangle = \frac{55}{2^3 3^{3/2}} P_{\gamma} \varepsilon_c \)

\[
8.14 \quad H(x_p,x_p^{'}) \frac{55}{2^3 3^{3/2}} \frac{P_{\gamma} \varepsilon_c}{P_c^2} \delta s \quad \text{← last term}
\]

Here, \( H(x_p,x_p^{'}) \) is the CSL invariant form with the dispersion \( x_p \) and \( x_p^{'} \) substituted.
Finally the equation for $J_x$ is

$$8.15 \quad \dot{J}_x = -DE\left\langle B^2 \right\rangle - \left\langle (2BB' + B^2\Omega)x_p \right\rangle J_x + H(x_p, x'_p) \frac{55}{2}\frac{P_y \varepsilon_c}{E_c}$$

In combined function accelerators the BB' term turns out to be about twice the $B^2$ term, so $J_x$ grows. In separated function accelerators either $B$ or $B'$ is zero, so the BB' term is zero. The other antidamping term is related to the momentum compaction and is usually small, except in weak focusing structures. Under these conditions, $J_x$ damps or grows to an equilibrium value given by the ratio of the diffusion term to the damping rate.

In the phase oscillations the change of $W_E$ is determined by the change in energy alone, since the time coordinate does not change

$$8.16 \quad \delta W_E = \frac{-2\Delta Ec\delta P + (c\delta P)^2}{Y}$$

vertical ellipses because for phase oscillations CSL $\alpha_E=0$
again the linear term $\Delta E$ in the phase oscillation has an average because of the energy dependence through the spatial dependence of the energy loss as well as the direct energy dependence. See Eq. 8.2 & 8.6.

\[ 2\langle (\Delta E)^2 \rangle = Y W_E \]  

(similar to 8.12)

\[ \dot{W}_E = \frac{-2YW_E}{2Y} \Delta E \langle 2B^2 + x_p (2BB' + B^2\Omega) \rangle + \frac{c^2\langle (\delta P)^2 \rangle}{Y} \]

8.17

\[ = -\alpha_E W_E + \frac{55}{2^{33/2}} \frac{P_\gamma e_c}{Y} \]

\[ \alpha_E = \Delta E \langle 2B^2 + x_p (2BB' + B^2\Omega) \rangle \]

It is interesting to note that the equilibrium mean square energy spread

\[ \langle (\Delta E)^2 \rangle = \frac{Y\langle W_E \rangle_{eq}}{2} = \frac{55}{2^{33/2}} \frac{P_\gamma e_c}{2\alpha_E} \]

is independent of the RF frequency, voltage, etc., and that the horizontal beam size is proportional to Planck's constant.
Let us now look at the "natural" size of the beam due to the variation in photon emission angle. We can calculate the equilibrium vertical beam size due to this effect.

\[ \delta J_z = \beta_L P \left( \frac{\varepsilon \psi}{P_c} \right)^2 = \beta_L \left\langle \frac{\varepsilon^2}{P_c^2} \right\rangle \frac{\delta t}{P_c^2} \]

8.19

\[ \dot{J}_z = \frac{\beta_L}{2^{3/2}} \frac{55}{\gamma^2 P_c^2} \frac{1}{\gamma^2 P_c^2} \]

(\( \beta_L \) is the lattice beta function.) The ratio of this term to the similar term in \( x \) is \( \beta_L/H(x_p, x'_p) \gamma^2 \) which is very small. In fact, the ability to measure the beam height is limited by diffraction. Attempts to exploit the small beam size to achieve high luminosity in colliders are frustrated by the beam-beam (space charge) effect. A better result is achieved if the beam is made larger by coupling \( x \) and \( z \) and operating with higher current in each beam.
The beam size is then approximately \( \sqrt{\frac{\beta L H(x_p, x'_p)}{E}} \), and will increase proportional to energy unless the lattice is modified to reduce \( \beta_L \) and \( H \). This can be done simply by shortening the cell length and increasing the tune, or by specially modifying the lattice to reduce the dispersion in the bend magnets. The first method was used in the SLAC damping rings for the Stanford Linear Collider (SLC), whereas the second method has been used in the Advanced Light Source at Berkeley and the Advanced Photon Source at Argonne.

Synchrotron Radiation has been used to help achieve collisions in e\(^+\)-e\(^-\) colliders from 200 MeV at Novosibirsk to 200 GeV at the Large European Project (LEP) at CERN. For higher energies, the penalties paid in power are so great that Linear Colliders will be employed for e\(^+\)-e\(^-\) collisions. This will be the subject of another lecture. As an alternative to e\(^+\)-e\(^-\) colliders, \( \mu^+\)-\( \mu^-\) colliders are under study.
Synchrotron radiation has provided excellent spectral quality for intense photon beams for investigation in a variety of disciplines of science. These include Chemistry, Surface Physics, Condensed Matter Physics, Biology, and Medicine. Recently the advent of intense synchrotron x-ray light sources has opened up new areas to exploitation.

Another means of exploiting synchrotron radiation, rather than using the radiation from bending magnets, is to employ "wigglers" in a straight section of a storage ring. A wiggler is a magnet that gives no net bending or offset, but has a periodically varying field with a small wavelength $\lambda_{\text{wiggler}}$ (with perhaps higher harmonics of field also). In the electron frame, the oscillation frequency of the electron is increased by a factor of $\gamma$, and the field increased by the same factor. The electron radiates at that frequency and the radiation is Lorentz boosted in frequency by another factor of $\gamma$ to the laboratory, where it is contained in a cone of opening angle $1/\gamma$. 
The peak emission is at a wavelength $\lambda = \frac{\lambda_{\text{wiggler}}}{2\gamma^2}$. The bandwidth of the radiation varies inversely with the number of wiggler periods, so it can have a rather narrow bandwidth, something not available with bending magnet radiation. The total power from a wiggler can be calculated classically, as in bending magnet radiation.
End of Lecture