Lecture 5: Lattice Calculations
Summary

A. Include dispersion in transfer matrix (3x3 matrix $T$)
B. Dispersion vector and phase advance
C. $\alpha, \beta, \gamma$ transfer matrix
D. IDA – beamline and accelerator design program
Solutions for Lattices

The design of an accelerator must incorporate, at a very early time, an understanding of the properties of the orbits and the elements needed to contain and accelerate the beam. This is greatly facilitated by a knowledge of the linear properties of the orbits, including tunes, beta functions, dispersion and a knowledge of the beam properties.

To describe completely the motion, we need to include the dispersion, as in 3.16 and 3.17,

\[ x'' + \frac{P_0}{P} (K + \Omega^2) x = \Omega \frac{\Delta P}{P}, \quad z'' - \frac{P_0}{P} K z = 0 \]

\[ x = X + x_p(s) \frac{\Delta P}{P} \quad x_p'' + (K + \Omega^2) x_p = \Omega \quad X'' + \frac{P_0}{P} (K + \Omega^2) X = 0 \]

However, since the motion is in the x-coordinate only, we will use “\( K \)” to represent “\((K+\Omega^2)\)”. We do this by augmenting the matrix \( M \) to include the momentum offset \( \delta = \frac{\Delta P}{P} \) so that \( M \) becomes a three by three matrix \( T \). The 13 and 23 terms contain the particular integral of 3.17 and its derivative, while the 33 term equals one, since the momentum is unchanged in a magnet.
Definition: **singular points** in differential equation \[ y''(x) + P(x)y' + Q(x)y = 0 \] (Arfken p.387)

If \( P(x) \) and \( Q(x) \) remain finite at \( x=x_0 \) then \( x_0 \) is an ordinary point. If either \( P(x) \) or \( Q(x) \) (or both) diverge as \( x \to x_0 \) then \( x_0 \) is a singular point.

1. If either \( P(x) \) or \( Q(x) \) diverges as \( x \to x_0 \) but \( (x-x_0)P(x) \) and \( (x-x_0)^2Q(x) \) remain finite as \( x \to x_0 \), then \( x=x_0 \) is called regular or nonessential singular point.

2. If \( P(x) \) diverges faster than \( 1/(x-x_0) \) so that \( (x-x_0)P(x) \) goes to infinity as \( x \to x_0 \), or \( Q(x) \) diverges faster than \( 1/(x-x_0)^2 \) then \( x=x_0 \) is an irregular or essential singularity.

3. Likewise there is an evaluation as \( x \to \infty \), etc.

A 2\(^{nd}\)-order diff. homogeneous eq. may have a power series solution. By Fuch’s theorem this is possible provided that the power series is an expansion around an ordinary point or a nonessential singularity.

Inhomogeneous eqn \[ y''(x) + P(x)y' + Q(x)y = F(x) \]. If \( y_p \) is a specific solution, we may add to it any solution of the corresponding homogeneous eqn.

Hence the most general solution is \[ y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x) \]
The 31 and 32 terms are zero since position and angle offsets do not induce momentum offsets. Then the matrix is

\[
\mathbf{T} = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
0 & 0 & 1
\end{bmatrix}
\]

which operates on the vector

\[
\mathbf{X} = \begin{bmatrix}
x \\
x' \\
\delta
\end{bmatrix}
\]

We write the solutions, solve for the final \( x \) and \( x' \) in terms of arbitrary constants, solve for the constants in terms of the initial \( x, x' \), and \( \delta \), and so get the final \( x \) and \( x' \) in terms of the initial \( x, x' \), and \( \delta \). The particular integrals are 0 for the drift, and \( \frac{\Omega \delta}{K} \ast \) for the magnets. We thus calculate \( \mathbf{T} \) for drifts, F magnets, and D magnets.

\[
\Omega \delta
\]

where \( K = K + \Omega^2 \)
A method for particular integrals on functions such as sine, cosine, polynomials, etc.

\[ x''_p + \left(K + \Omega^2\right)x_p = \Omega \]

\[
\left[ D^2 + \left(K + \Omega^2\right) \right] x_p = \Omega
\]

\[
x_p = \frac{\Omega}{D^2 + \left(K + \Omega^2\right)} = \frac{1}{\left(K + \Omega^2\right)} \left\{ \frac{1}{1 + \frac{D^2}{K + \Omega^2}} \right\} \Omega
\]

\[
x_p = \frac{1}{\left(K + \Omega^2\right)} \left(1 - \frac{D^2}{\left(K + \Omega^2\right)}\right) \Omega = \frac{1}{\left(K + \Omega^2\right)} (1) \Omega
\]

\[
x_p = \frac{\Omega}{K + \Omega^2}
\]

\( \Omega \) is a constant.
5.3 \[ T_d = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

5.4 \[ T_f = \begin{bmatrix} \cos \sqrt{K}s & \sin \sqrt{K}s & \Omega \left(1 - \cos \sqrt{K}s\right) \\ -\sqrt{K}s \sin \sqrt{K}s & \cos \sqrt{K}s & \frac{K}{\Omega \sin \sqrt{K}s} \\ 0 & 0 & 1 \end{bmatrix} \]

where \( K = K + \Omega^2 \)
If at either end/edge of a bend or gradient magnet, the surface is not in the x,z plane, but at an angle $\phi$ to the x,z plane (positive $\phi$ means that there is less bending on the positive x side), there is a focusing effect given by

$$T_D = \begin{bmatrix}
\cosh\sqrt{-K}s & \frac{\sinh\sqrt{-K}s}{\sqrt{-K}} & \Omega\left(1 - \cosh\sqrt{-K}s\right) \\
\sqrt{-K}\sinh\sqrt{-K}s & \cosh\sqrt{-K}s & \frac{\Omega\sinh\sqrt{-K}s}{\sqrt{-K}} \\
0 & 0 & 1
\end{bmatrix}$$

The sign of $T_{21}$ is opposite for the z motion ($\nabla_x B = 0$). The effect is called "edge focusing". *
edge focusing: bending magnet with magnetic field lines

When \( \mathbf{u} \) is not in the direction of \( \mathbf{s} \), the magnetic field has an \( x \) component.
To find the lattice parameters, we calculate the $T$ matrices for $x$ and $z$ for each lattice element (in which $s$ is the element length), and separately multiply the $x$ and $z$ matrices in the sequence in which they occur in the lattice period, obtaining the $x$ and $z$ one period matrices $T$ as in 4.15.

$$M(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{bmatrix} = I \cos \mu + J \sin \mu \tag{4.15}$$

Next we use equations 4.10 - 4.14 to find $\mu$, and if it is real, $\alpha$, $\beta$, and $\gamma$ from the 2 by 2 part of $T$. The dispersion vector $X_p$ is that vector which, when transformed by $T$, becomes $X_p$ (the closed orbit solutions). That is,

$$5.7 \quad TX = X \quad \text{or} \quad (T - I)X = 0$$

4.10-4.14

$$\cos \mu = \frac{1}{2} \text{Tr} M = \frac{a + d}{2} \quad \lambda = \cos \mu \pm i \sin \mu \quad |a + d| \leq 2$$

$$\begin{align*}
\frac{a - d}{2} &= 2\alpha \sin \mu \\
\frac{b}{2} &= \beta \sin \mu \\
\frac{c}{2} &= -\gamma \sin \mu \\
\gamma &= \frac{1 + \alpha^2}{\beta}
\end{align*}$$
We can decompose $T$ into a 2X2 matrix, a 2X1 column vector $v$, a 1X2 zero row vector $0$, and 1X1 unit matrix $I$

$$5.8 \quad T = \begin{bmatrix} m & v \\ 0 & 1 \end{bmatrix}, \quad m = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad v = \begin{bmatrix} M_{13} \\ M_{23} \end{bmatrix}, \quad x = \begin{bmatrix} x \\ x' \end{bmatrix}$$

The existence of a solution to 5.7 requires that the $\det(T-I) = 0$, which yields

$$5.9 \quad (m - I)x + v\delta = 0$$

where $\delta = \frac{\Delta P}{P}$ and (I is the 2X2 identity)

Since

$$5.10 \quad \det(m - I) = 2 - \text{Tr}(m) = 2 - 2\cos \mu$$

the matrix $m-I$ has an inverse except in uninteresting cases. We find

$$5.11 \quad (m - I)^{-1} = \frac{1}{2} \begin{bmatrix} M_{22} - 1 & -M_{12} \\ -M_{21} & M_{11} - 1 \end{bmatrix}$$

*see notes*
Determinants: for a 2x2 matrix A

\[
\text{Let } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ then } A^{-1} = \frac{1}{\det |A|} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}
\]

Properties of matrices and determinants

a matrix A is said to be singular if \( \det |A| = 0 \)

\[
\det(AB) = \det(A)\det(B) = \det(BA)
\]

\[
\text{tr}(AB) = \sum_i (AB)_{ii} = \sum_{i=1}^n \sum_{l=1}^n a_{il} b_{li} = \sum_{l=1}^n \sum_{i=1}^n b_{li} a_{il} = \sum_j (BA)_{jj} = \text{tr}(BA)
\]
5.9 becomes

\[
\begin{align*}
\begin{bmatrix}
    x \\
    x'
\end{bmatrix}_p &= \frac{\delta}{2 - 2\cos\mu} \begin{bmatrix}
    M_{22} - 1 & -M_{12} \\
    -M_{21} & M_{11} - 1
\end{bmatrix} \begin{bmatrix}
    M_{13} \\
    M_{23}
\end{bmatrix} = \frac{\delta}{2 - 2\cos\mu} \begin{bmatrix}
    M_{13} (M_{22} - 1) - M_{12} M_{23} \\
    -M_{21} M_{13} + M_{23} (M_{11} - 1)
\end{bmatrix}
\end{align*}
\]

This is the dispersion vector at the starting point. It, and \( M \) (or \( m \)) (part of \( T \)) can be transformed successively by the \( T \) matrices to find the dispersion vector and the lattice functions at all points in the calculation. The phase advances between each two points can be found by inspection of 4.31* as applied to each \( T \) matrix. The phase advance \( \psi_{12} \) between points 1 and 2 is given by

\[
\tan \psi_{12} = \frac{T_{12}}{T_{11}\beta_1 - T_{12}\alpha_1}
\]

*see notes

In this way all lattice functions and the phase advance can be specified at each point in the lattice.
\[ M(s_2|s_1) = \begin{bmatrix} \sqrt{\beta_2} (\cos \psi + \alpha_1 \sin \psi) & \sqrt{\beta_1 \beta_2} \sin \psi \\ \cos \psi \left( \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \right) - \sin \psi \left( \frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \right) & \sqrt{\beta_1} (\cos \psi - \alpha_2 \sin \psi) \end{bmatrix} \]

\[
\tan \psi_{12} = \frac{T_{12}}{T_{11} \beta_1 - T_{12} \alpha_1}
\]

\[
\tan \psi_{12} = \frac{\sin \psi \sqrt{\beta_1 \beta_2}}{\beta_1 \left[ \sqrt{\beta_2} \cos \psi + \alpha_1 \sqrt{\beta_2} \sin \psi \right] - \alpha_1 \sqrt{\beta_1 \beta_2} \sin \psi} = \frac{\sin \psi \sqrt{\beta_1 \beta_2}}{\sqrt{\beta_1 \beta_2} \cos \psi + \alpha_1 \sqrt{\beta_1 \beta_2} \sin \psi - \alpha_1 \sqrt{\beta_1 \beta_2} \sin \psi} = \tan \psi_{12}
\]
We need to find the momentum compaction \( \alpha \) of 3.19* above, and the chromaticity described near 3.15* above. We can use 5.4 or 5.5 to find the matrix elements inside the bend magnets to find the dependence of \( \Omega x_p \) on \( s \) and the initial values of the dispersion vector. Performing the integration, we find

\[
\int_0^L \Omega x_p \, ds = \frac{\Omega}{K} \left( \Omega L - x'_p + x''_p \right)
\]

where \( K = K + \Omega^2 \)

that is, the bending angle minus the change in slope of the dispersion function. We add up the value for all bends, and divide by the orbit length to find \( \alpha \).

For the chromaticity, we note that the effect of a momentum offset on the betatron oscillations is an added focusing term

\[
\Delta K = -K \frac{\Delta P}{P} \Rightarrow \sim \quad x'' + \left( K + \Omega^2 \right) \left( 1 - \frac{\Delta P}{P} \right) x = 0
\]

see eqn 3.17
Near 3.15:

\[ x'' + \frac{P_0}{P} (K + \Omega^2) x = \Omega \frac{\Delta P}{P} \]

This is called "chromaticity".

Further the frequencies of the oscillations will depend on momentum. This is called "chromaticity".

Note that the phase advance is \( \mu = \cos^{-1}\left[\frac{1}{2}\text{Tr}(M)\right] \), and the "frequency of betatron oscillations" is called the tune and is \( \nu = N \mu / 2\pi \) which is the number of betatron oscillations per revolution.
We will learn later that the effect of a gradient error on tune $\nu$ is given by an approximate relation

$$\Delta \nu = \frac{1}{4\pi} \int \beta \Delta K ds$$

That is, we need to integrate the $\beta$ functions through the focusing elements. To do that we can use an extremely useful result from the literature relating $\alpha$, $\beta$, and $\gamma$ in an element to their initial values by a matrix transformation

$$\begin{bmatrix}
\beta \\
\alpha \\
\gamma
\end{bmatrix}_2 =
\begin{bmatrix}
T_{11}^2 & -2T_{11}T_{12} & T_{12}^2 \\
-T_{11}T_{21} & 1 + 2T_{21}T_{12} & -T_{22}T_{12} \\
T_{21}^2 & -2T_{21}T_{22} & T_{22}^2
\end{bmatrix} \begin{bmatrix}
\beta \\
\alpha \\
\gamma
\end{bmatrix}_1$$

* reference (CERN/MPS-SI/Int. DL/70/4, Bovet, Gouiran, Gumowski, Reich)
This allows the integration of 5.16 to be performed. The quantities

\[ \xi = \frac{\Delta v}{\Delta P} \]

are calculated for x and z motion.

Often the sector will have a point of symmetry, and we can use this to simplify the calculation. Suppose \( M(b|a) \) is the half sector matrix. The matrix from b to the sector end is \( M^{-1}(b|a) \) with the sign of s reversed, since the particle is going in the opposite direction. This is the “reflect” matrix. We can easily get the inverse

\[
M^{-1} = \begin{bmatrix}
M_{22} & -M_{12} & M_{12}M_{23} - M_{22}M_{13} \\
-M_{21} & M_{11} & -M_{11}M_{23} + M_{21}M_{13} \\
0 & 0 & 1
\end{bmatrix}
\]
Next we change sign of $s$ to get the "Reflect" matrix by

$$5.20 \quad M^R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} M_{22} & M_{12} & M_{12}M_{23} - M_{22}M_{13} \\ M_{21} & M_{11} & M_{11}M_{23} - M_{21}M_{13} \\ 0 & 0 & 1 \end{bmatrix}$$

The full sector matrix is then

$$5.21 \quad M^S = M^R M = \begin{bmatrix} M_{11}M_{22} + M_{12}M_{21} & 2M_{12}M_{22} & 2M_{12}M_{23} \\ 2M_{21}M_{11} & M_{11}M_{22} + M_{12}M_{21} & 2M_{11}M_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

We note that $M_{11}^S = M_{22}^S = 1 + 2M_{12}M_{21}$. (use $M_{11}M_{22} - M_{21}M_{12} = 1$)
Fortunately, there are a number of programs available to perform these and other calculations. We will give examples of the use of a program called IDA (Interactive Design of Accelerators) written by Mark Barton of Brookhaven Laboratory and IBM Corporation. The main outlook of the program is to design electron storage rings for Synchrotron Light Sources, however it can be used for most types of lattices or beam transport lines. It operates in a DOS (PC) environment. Some of the display pages are shown below. The first is a title page and list of directories with files for different accelerators.

IDA BEAMLINE AND ACCELERATOR DESIGN PROGRAM

Mark Q. Barton

Version IV.2, February 4, 1991

….etc.
A number of other programs have been written, for example SYNCH by Alpert Garren of LBL in Berkeley, CA, and MAD by Everhard Keil of CERN at Genève, CH. These programs include possibilities for more advanced problems. IDA is convenient for this use since it performs all the elementary calculations, yet is small enough to fit on one floppy disk. Also included is a program called ELISE, which calculated some quantities important to collective instabilities in Light Sources.
End of Lecture