Lecture 2: Pendulum Motion
Summary

A. The pendulum equation
   - small amplitude motion
   - large amplitude motion
B. The biased pendulum equation
Pendulum Motion

A number of systems we shall investigate follow the same equations as the simple pendulum. As an example of the methods we will apply, consider the motion of a physical pendulum, whose moment of inertia is $I$ around its support point. The distance from the support to the center of mass of the pendulum is $d$. Let $\theta$ be the angle between the vertical and the line from the support to the center of mass.
Then

2.1 \[ T = I \frac{\dot{\theta}^2}{2}, \quad V = -mgd \cos \theta \] I - moment of inertia

2.2 \[ L = T - V = I \frac{\dot{\theta}^2}{2} + mgd \cos \theta \]

2.3 \[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta} \]

2.4 \[ H = p_\theta \dot{\theta} - L = \frac{p_\theta^2}{2I} - mgd \cos \theta \]

Since \( H \) is not a function of \( t \), it is a constant of the motion, so the problem is formally soluble by quadratures. Let us first apply some approximate methods. When \( \theta << 1 \), we can approximate \( \cos \theta \approx 1 - \frac{\theta^2}{2} \). Then

2.5 \[ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgd\theta \]
so that

\[ 2.6 \quad \ddot{\theta} = -\omega^2 \theta \quad , \quad \omega = \sqrt{\frac{mg}{I}} \]

\( \omega \) is the (angular) frequency of small amplitude motion. Now consider large amplitude motion. Let us start the pendulum at rest at angle \( \theta_0 \). Then the value of \( H \) is \(-\omega^2 I \cos \theta_0\). Solve for

\[ 2.7 \quad p_\theta = I \omega \sqrt{2(\cos \theta - \cos \theta_0)} \quad *\text{see notes} \]

and plot the orbits \( p_\theta \) versus \( \theta \), as in Fig.2. There are fixed (equilibrium) points at \( p_\theta = 0, \theta = 0 \) and \( \theta = \pm \pi \). The fixed point at \( \theta = 0 \) is stable, and the small amplitude orbits are ellipses around the origin. The orbit which passes through the unstable fixed points at \( \theta = \pm \pi \) is called the "separatrix". It divides the phase plane into two regions, a stable (vibration) region inside the separatrix (this is called the "bucket", where one hopes to keep the particles), and an unstable (libration) region outside the separatrix.
derivation of $p_\theta = I\omega\sqrt{2(\cos \theta_0 - \cos \theta)}$

Pendulum starts at rest and $H$ is conserved, i.e. $H = H_0$.

$$H = \frac{p_\theta^2}{2I} - mgd \cos \theta = -mgd \cos \theta_0$$

$$\omega^2 = \frac{mgd}{I}$$

and drop mgd constants.

$$p_\theta^2 = 2\omega^2 I^2 (\cos \theta - \cos \theta_0)$$

and

$$p_\theta = I\omega\sqrt{2(\cos \theta - \cos \theta_0)}$$

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The maximum excursion of $p_\theta$ is

$$2.8 \quad p_{\theta,\text{max}} = 2I\omega$$

Now we can use the energy integral to calculate several properties of the orbits. The period $T$ of the orbit is

$$2.9 \quad T = \oint dt = \oint \frac{d\theta}{\dot{\theta}} = \oint \frac{d\theta}{\omega \sqrt{2(\cos \theta - \cos \theta_0)}} = \left( \oint I \frac{d\theta}{p_\theta} \right)$$

The area of the orbit in phase space is

$$2.10 \quad A = \oint p_\theta d\theta = I\omega \oint \sqrt{2(\cos \theta - \cos \theta_0)} d\theta$$
The total area of the stable (vibration) region is given for $\theta_0 = \pm \pi$

2.11 \[ A = 16I\omega \]

For $\theta_0 \neq \pm \pi$, these are elliptic integrals and can be found in tables. On the other hand, they are trivial integrals to perform numerically.

As a further example of the methods we will employ, let us transform to angle action variables $(J, \gamma)$ for the linearized problem described above. The generating function is

2.12 \[ F_1 = \frac{I\omega \theta^2 \cot \gamma}{2} \]

and the transformation equations are

2.13 \[ J = \frac{p_{\theta}^2 + (I\omega \theta)^2}{2I\omega}, \quad \cot \gamma = \frac{p_{\theta}}{I\omega \theta} \]

2.14 \[ p_{\theta} = \sqrt{2I\omega J} \cos \gamma, \quad \theta = \sqrt{\frac{2J}{I\omega}} \sin \gamma \]

*see notes*
derivation of 2.13 and 2.14.

\[ p_i = \frac{\partial F_i}{\partial q_i}, \quad p_j = -\frac{\partial F_i}{\partial Q_j}, \quad K = H + \frac{\partial F}{\partial t} \]

\[ p_i = \frac{\partial}{\partial \theta} \left( \frac{\omega I \theta^2 \cot \gamma}{2} \right) = I \theta \cot \gamma \quad \text{so that} \quad \cot \gamma = \frac{p_\theta}{I \theta} \]

\[ P_\gamma = -\frac{\partial}{\partial \gamma} \left[ \frac{\omega I \theta^2 \cot \gamma}{2} \right] = \frac{\omega I \theta^2}{2} \left[ 1 + \cot^2 \gamma \right] = \frac{\omega I \theta^2}{2 \sin^2 \gamma} = J \]

\[ P_\gamma = \left[ \frac{\omega I \theta^2}{2} + \frac{p_\theta^2}{2I \omega} \right] = \frac{p_\theta^2 + (I \omega \theta)^2}{2I \omega} = J \]

then \[ \theta = \sqrt{\frac{2J}{I \omega}} \sin \gamma \quad \text{and} \quad p_\theta = \sqrt{2I \omega J} \cos \gamma \]
The Hamiltonian is

\[ H = \omega J + \text{(higher order terms)} \] *see notes

which come from the expansion of \( \cos \theta \). Keeping terms through fourth order, and dropping constants and functions of time only,

\[ H = \omega J - \frac{J^2 \sin^4 \gamma}{6I} \] *see notes

Averaging over \( \gamma \),

\[ \langle H \rangle = \omega J - \frac{J^2}{16I} \]

The mean angular frequency is

\[ \dot{\gamma} = \omega - \frac{J}{8I}, \quad \frac{\delta \omega}{\omega} = - \frac{J}{8I \omega} \] from \( \dot{\gamma} = \frac{\partial H}{\partial J} \)
\[ H = \frac{p_\theta^2}{2I} - I\omega^2 \cos \theta = \frac{\cos^2 \gamma 2J I\omega}{2I} - I\omega^2 \left(1 - \frac{\theta^2}{2} + \text{h.o.t.}\right) \]

\[ A = -I\omega^2 \left(1 - \frac{\sin^2 \gamma 2J}{2I\omega}\right) = -I\omega^2 + J\omega \sin^2 \gamma \]

\[ H = (\cos^2 \gamma) J\omega + (\sin^2 \gamma) J\omega - I\omega^2 \]

drop constants and we get \[ H = J\omega \]

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going to fourth order in \( \theta \), \( A \) becomes

\[ A = -I\omega^2 \left(1 - \frac{J \sin^2 \gamma}{I\omega} + \frac{J^2 \sin^4 \gamma}{6I^2 \omega^2}\right) = -I\omega^2 + J\omega \sin^2 \gamma - \frac{J^2 \sin^4 \gamma}{6I} \]

\[ H = J\omega - \frac{J^2 \sin^4 \gamma}{6I} - I\omega^2 \]

and again dropping constants we get \[ H = J\omega - \frac{J^2 \sin^4 \gamma}{6I} \]

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The area of an orbit is $2\pi J$, so that when $2\pi J \approx 16I\omega$, we are about at the edge of the stable oscillation region, so that our approximation says \[
\frac{\delta \omega}{\omega} = -\frac{1}{\pi} \]

In reality the separatrix orbit has zero frequency, because the pendulum can sit forever on the unstable fixed point if there are no perturbations. The actual frequency vs. area is shown in Fig. 2.3 (normalized variables).

$$\Omega = \frac{T_0}{T}$$

Action = $2\pi J/16I\omega$
A second variant of pendulum is the biased pendulum, as seen in Figure 2.4.

A massless pulley has a mass \( m \) attached on its perimeter of radius \( d \). A mass \( M \) is suspended by a massless cord wrapped around and attached to the pulley. The equilibrium is no longer at \( \theta = 0 \), but is shifted to some value \( \theta_s \) where the torque \( Mgd \) exerted on the pulley by the weight of mass \( M \) is balanced by the torque \(-mgds\sin\theta\) exerted on the pulley by the weight of mass \( m \) attached to it. Evidently, \( \sin\theta_s = \frac{M}{m} \). The unstable equilibrium is now at \( \theta = \pi - \theta_s \). Then

\[
2.19 \quad T = I\frac{\dot{\theta}^2}{2}, \quad V = -mgd\cos\theta - Mgd\theta = -mgd(\cos\theta + \Gamma\theta)
\]

\( \text{(not differentials!)} \)

\[
2.20 \quad I = I_{\text{pulley}} + (M + m)d^2, \quad \Gamma = \sin\theta_s
\]

\[
2.21 \quad H = \frac{p^2_{\theta}}{2I} - mgd(\cos\theta + \Gamma\theta)
\]
Now we can transform to $P_\theta = p_\theta$, $\tau = \theta - \theta_s$ via $F_2 = P_\theta(\theta-\theta_s)$, where in general $\theta_s$ can be a function of time. The new Hamiltonian

$$
2.22 \quad K = \frac{(P_\theta - I\dot{\theta}_s)^2}{2I} - mgd[\cos(\tau - \theta_s) + \Gamma(\tau + \theta_s)]
$$

where we have ignored constants and functions only of time. We see that if $\dot{\theta}_s \neq 0$, the equilibrium occurs at non-zero $P_\theta$, at $I\dot{\theta}_s$. In what follows, we will consider only cases where $\dot{\theta}_s = 0$ (i.e. $K = \text{constant}$). Again we can expand the cosine function for small values of $\tau$, and find the frequency of small amplitude oscillations to be

$$
2.23 \quad \omega = \sqrt{\frac{mgd \cos \theta_s}{I}}
$$

We can solve for $P_\theta$ as a function of $\tau$ (or as here $\theta$) and plot the phase space trajectories, as in Figure 2.5, where $\Gamma = 0.5$. 
We can see immediately that the width of the stable region, both in $P_\theta$ and $\theta$ (the "bucket") is less than that in Figure 2.2. Figure 2.5 is called a "moving bucket" for reasons to be discussed later when we treat particle acceleration, and Figure 2.2 is called a "stationary bucket". The maximum value of $P_\theta$ on the boundary of the stable region is reduced to

$$2.24 P_{\theta,\text{max}} = 2I\omega \left[ 1 - \left( \frac{\pi}{2} - \theta_s \right) \tan \theta_s \right]^{\frac{1}{2}}$$

where $\omega$ is that defined for the biased pendulum case.

*see notes
Notes: to solve for $P_{\theta_{\text{max}}}$ evaluate the constant Hamiltonian $K$ on the separatrix, then solve for $p_{\theta_{\text{max}}}$ and find the maximum value.
End of Lecture
derivations for Lecture 2: Pendulum Motion